TOWARDS A CHARACTERIZATION OF THE COVERT CAPACITY OF BOSONIC CHANNELS UNDER TRACE DISTANCE

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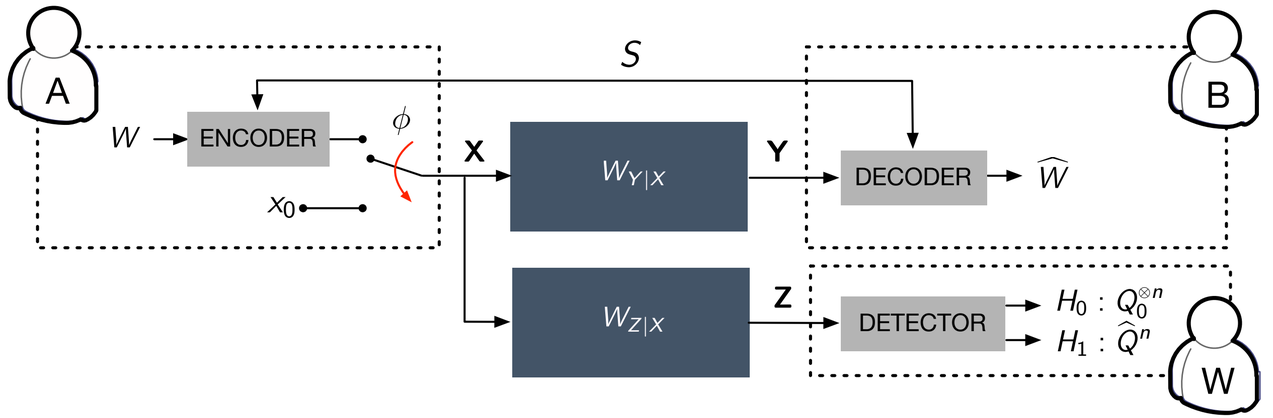
Shi-Yuan Wang, Tuna Erdoğan, and Matthieu R. Bloch

School of Electrical and Computer Engineering

- **COVERT COMMUNICATIONS**
  - Enhance privacy by **avoiding exposure of information transmission**
  - Communication with **low probability of detection**

- **FUNDAMENTAL LIMITS OF COVERT COMMUNICATIONS**
  - (*Square Root Law* **[Bash et al.'13]**) No more than $\mathcal{O}(\sqrt{n})$ bits over n uses of noisy memoryless channel
  - **Zero-rate** regime: rate of communication **vanishes** asymptotically

- **EXTENSIONS AND RELATED WORKS**
  - First- and second-order asymptotics **[Wang et al.'16, Bloch'16, Tahmasbi-Bloch'19]**
  - Multiuser channels **[Arumugam-Bloch'18'19'19, Tan-Lee'19,Cho-Li'21]**
  - Codes for covert communications **[Frèche et al.'17, Kadampot et al.'18'19, Lamarca-Matas'19, Zhang et al.'20, Wang-Bloch'21]**
  - AWGN channels **[Wang et al.'16, Zhang et al.'19, Yan et al.'19]**
  - MIMO-AWGN channels **[Abdelaziz-Koksal'17, Bendary et al.'19, Wang-Bloch'21]**
  - Variational distance constraint **[Tahmasbi-Bloch'19, Zhang et al.'19, Wang-Bloch'21]**
  - Classical-Quantum channels **[Bash et al.'15, Sheikholeslami et al.'16, Wang'16]**
  - Bosonic Channels **[Bullock et al.'20, Gagatsos et al.'20]**



- Switch $\phi \in \{0, 1\}$ controls transmission state at Alice
- **Innocent symbol** x_0 **:** *absence of communications*
  - Induces distributions $P_0 \triangleq W_{Y|X=x_0}$ and $Q_0 \triangleq W_{Z|X=x_0}$
- **Code** $\mathcal{C}$ **:** *occurrence of communications*
  - induces distribution $\widehat{Q}^n$ at Willie

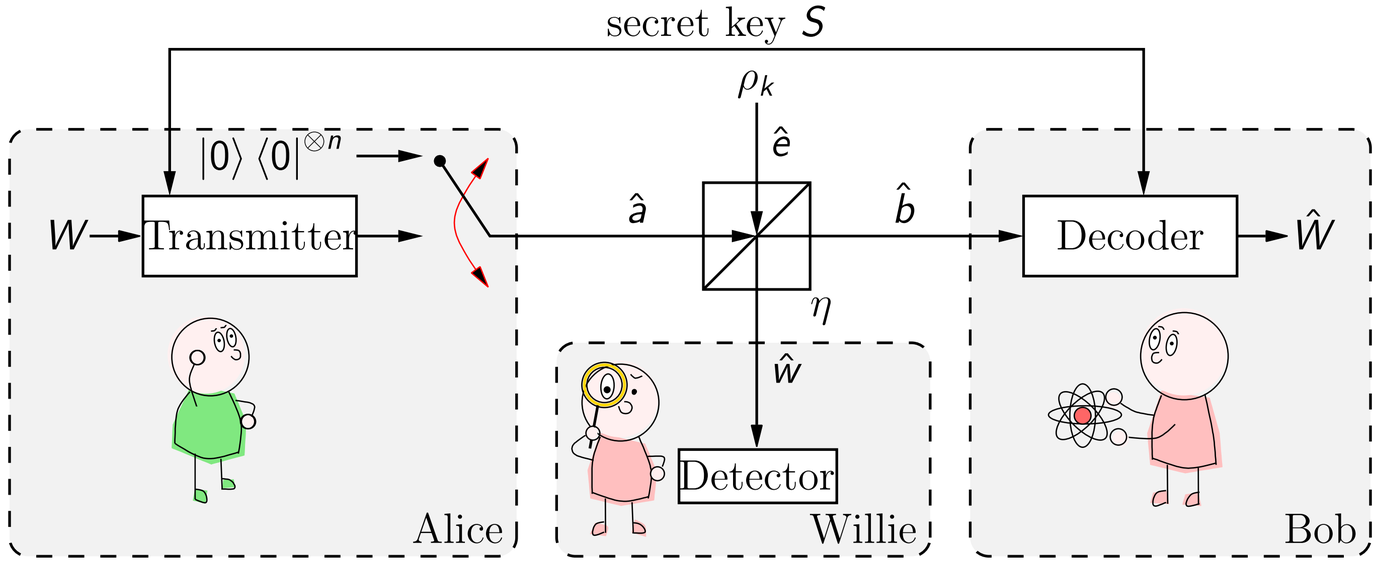
- **RELIABILITY:** Bob reliably recovers the message and guesses when the communication occurs

- **DETECTION:** Willie (passive warden) distinguishes
  - Hypothesis H_0 : $Q_0^{\otimes n}$ (*absence of communications*)
  - Hypothesis H_1 : $\widehat{Q}^n$ (*occurrence of communications*)

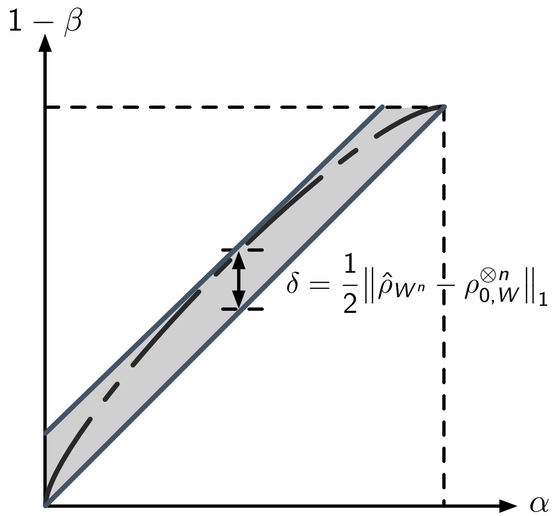
- **GOAL:** make sure Willie's test close to **blind test**

- Single-mode lossy thermal-noise bosonic channel $\mathcal{L}_{A\to BW}^{(\eta,k)}$
  - Transmissivity η
  - Described by a beamsplitter $\hat{b} = \sqrt{\eta}\hat{a} + \sqrt{1-\eta}\hat{e}$, and $\hat{w} = \sqrt{1-\eta}\hat{a} + \sqrt{\eta}\hat{e}$
  - Environment in thermal bath with mean photon number k
    - Thermal state

$$\rho_k = \frac{1}{\pi k}\int \exp\left(-\frac{|\alpha|^2}{k}\right) d^2\alpha\, |\alpha\rangle\langle\alpha| = \sum_{n=0}^{\infty} \frac{k^n}{(k+1)^{n+1}} |n\rangle\langle n|$$



- **DEFINITION:** $(M, K, n, \epsilon, \delta)$ -code $\mathcal{C}$
  - Uniformly distributed message $W \in [\![1, M]\!]$
  - Uniformly distributed secret key $S \in [\![1, K]\!]$ shared with Bob
  - Encoding channel $\mathcal{E}_{SW\to A^n} : |s\rangle\langle s|_S \otimes |w\rangle\langle w|_W \mapsto \rho_{A^n}(s, w)$
  - Collection of decoding POVMs $\{\{\Pi_{B^n}^{(s,w)}\}_{w\in[\![1,M]\!]}\}_{s\in[\![1,K]\!]}$
  - Induces $\hat{\rho}_{W^n} \triangleq \frac{1}{MK}\sum_{s=1}^{K}\sum_{w=1}^{M} \mathrm{tr}_{B^n}\left(\left(\mathcal{L}_{A\to BW}^{(\eta,k)}\right)^{\otimes n} \rho_{A^n}(s, w)\right)$ at Willie



- **RELIABILITY METRIC**
  - Maximal average probability of error $P_e \triangleq \max_{s\in[\![1,K]\!]} \mathbb{P}\left(\widehat{W} \neq W | S = s\right) \leqslant \epsilon$

- **COVERTNESS METRIC**
  - For any test $\mathcal{T}_{W^n\to\{0,1\}}$ conducted by Willie, the probabilities of false alarm α and missed detection β satisfy

$$1 \geqslant \alpha + \beta \geqslant 1 - \frac{1}{2}\left\|\hat{\rho}_{W^n} - \rho_{0,W}^{\otimes n}\right\|_1 \geqslant 1 - \delta$$

  - Any test conducted by Willie is close to **blind test**

- **Trace distance** is the covertness metric that carries operational meaning

- **[Tahmasbi-Bloch'19]** investigates the impact and operational meaning of different covertness metrics for classical case
  - Relative entropy $\mathbb{D}(\widehat{Q}^n \,\|\, Q_0^{\otimes n})$
  - Variational distance $\mathbb{V}(\widehat{Q}^n, Q_0^{\otimes n})$
  - Optimal probability of missed detection $\beta_\alpha(Q_0^{\otimes n}, \widehat{Q}^n)$

- **CHOICE OF COVERTNESS METRIC MATTERS**
  - **No strong converse** for covertness in covert communication
  - Asymptotics of covert capacity depend on covertness metric

- **DRAWBACKS OF RELATIVE ENTROPY**
  - Loose proxy for variational distance, more stringent
  - Operational meaning preserved in variational distance

- Using variational distance leads to a 25% relative increase in throughput **[Wang-Bloch'21]**

- **TAKE AWAY:** variational distance is more operationally relevant, and so is trace distance

- **KNOWN RELATIONS [Hayashi'17]**

$$1-\sqrt{F(\rho,\sigma)} \leqslant \frac{1}{2}\|\rho-\sigma\|_1 \leqslant \sqrt{1-F(\rho,\sigma)} \leqslant \sqrt{\mathbb{D}(\rho\,\|\,\sigma)}$$

  - Fidelity is multiplicative for tensor-product states
  - Fidelity retains properties of distance through purified distance $P(\rho,\sigma) \triangleq \sqrt{1-F(\rho,\sigma)}$

- **[Bullock et al.'20], [Gagatsos et al.'20]** use quantum relative entropy

- We use purified distance (and therefore fidelity) as intermediate metric in achievability
  - **TECHNICAL DETAIL:** require triangle inequality and purified distance for resolvability analysis

DEFINITION: ACHIEVABLE THROUGHPUT AND COVERT CAPACITY

$(M, K, n, \epsilon, \delta)$ - code is ϵ-reliable and δ-covertness if $P_e \leqslant \epsilon$ and $\frac{1}{2}\left\|\hat{\rho}_{W^n} - \rho_{0,W}^{\otimes n}\right\|_1 \leqslant \delta$.

The maximum number of messages that can be transmitted by an $(M, K, n, \epsilon, \delta)$-code is $M^*(n, \epsilon, \delta)$.

Covert Capacity:

$$C_{\text{cov}} \triangleq \lim_{n\to\infty} \frac{\log M^*(n, \epsilon, \delta)}{\sqrt{n}}$$

For a sequence of code achieving covert capacity, **secret key throughput** is $\lim_{n\to\infty} \frac{\log K}{\sqrt{n}}$.

THEOREM: COVERT CAPACITY OF LOSSY THERMAL-NOISE BOSONIC CHANNEL

Covert capacity satisfies

$$\frac{2\sqrt{\eta k(\eta k+1)}}{1-\eta}\eta\log\left(1+\frac{1}{(1-\eta)k}\right)Q^{-1}\left(\frac{1-\delta}{2}\right) \geqslant C_{\text{cov}} \geqslant \frac{2\sqrt{\eta k(\eta k+1)}}{1-\eta}\eta\log\left(1+\frac{1}{(1-\eta)k}\right)\delta$$

Lower bound is achievable with key throughput satisfying

$$2\sqrt{\eta k(\eta k+1)}\left(\log\left(1+\frac{1}{\eta k}\right)-\frac{\eta}{1-\eta}\log\left(1+\frac{1}{(1-\eta)k}\right)\right)^+\delta$$

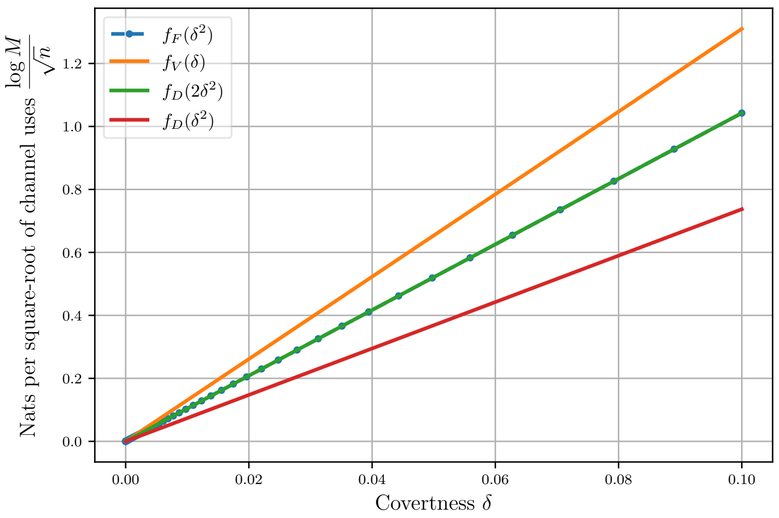
where $(x)^+ \triangleq \max(x, 0)$.

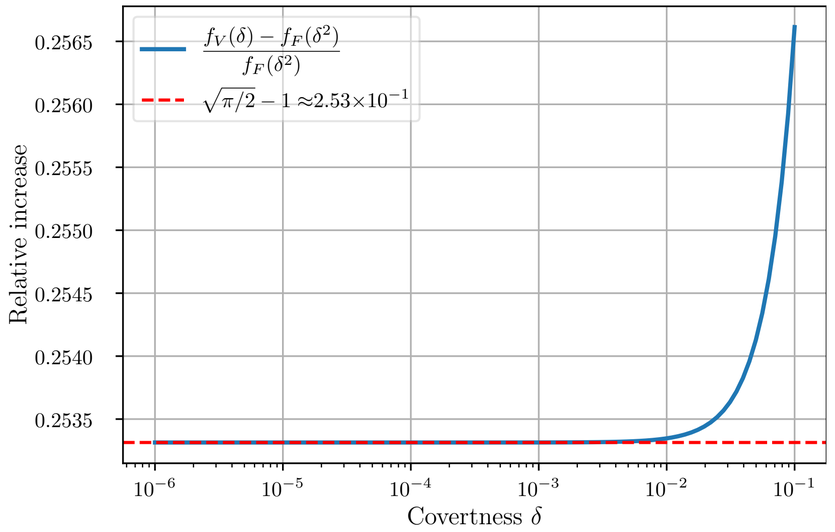
- **TAKE AWAY:** covert capacity is a function of covertness metric.

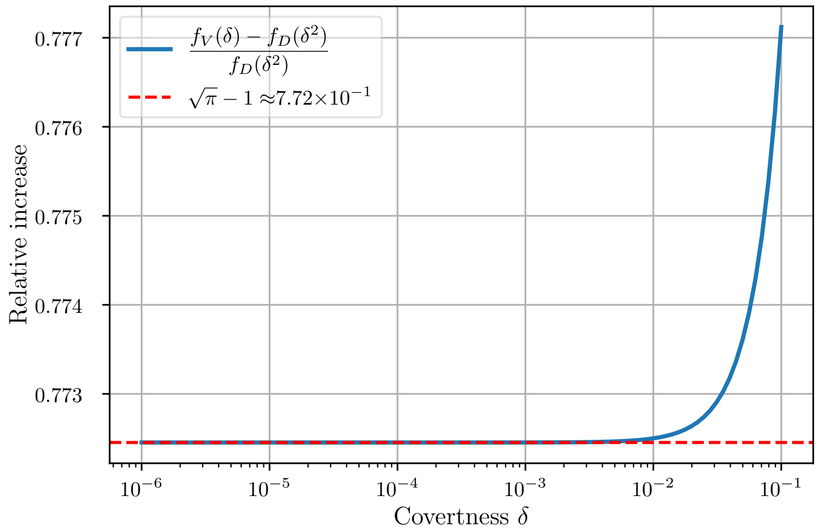
- **COMPARISON:** given $\frac{1}{2}\left\|\hat{\rho}_{W^n} - \rho_{0,W}^{\otimes n}\right\|_1 \leqslant \sqrt{1 - F(\hat{\rho}_{W^n}, \rho_{0,W}^{\otimes n})} \leqslant \sqrt{\mathbb{D}(\hat{\rho}_{W^n} \,\|\, \rho_{0,W}^{\otimes n})} \leqslant \delta$
  - Upper-bound is derived with trace distance metric, denoted by $f_V(\delta)$
  - Lower-bound is derived with fidelity metric, denoted by $f_F(\delta^2)$
  - We also compare previous result by **[Gagatsos et al.'20]** with quantum relative entropy metric, denoted by $f_D(\delta^2)$
- However, **[Bullock et al.'20]** and **[Gagatsos et al.'20]** consider **quantum Pinsker's inequality**
  - We denote this result by $f_D(2\delta^2)$

$$\frac{1}{2}\left\|\hat{\rho}_{W^n} - \rho_{0,W}^{\otimes n}\right\|_1 \leqslant \sqrt{\frac{\mathbb{D}(\hat{\rho}_{W^n} \,\|\, \rho_{0,W}^{\otimes n})}{2}} \leqslant \delta$$

- From above inequalities, $f_V(\delta) \geqslant f_F(\delta^2) = f_D(2\delta^2) \geqslant f_D(\delta^2)$







- **GOAL:** use **fidelity** to ensure $\frac{1}{2}\left\|\hat{\rho}_{W^n} - \rho_{0,W}^{\otimes n}\right\|_1 \leqslant \sqrt{1 - F(\hat{\rho}_{W^n}, \rho_{0,W}^{\otimes n})} \leqslant \delta$

- Coherent-state QPSK signaling with mean photon number decreasing in $\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$
  - Ensure signal to be close to innocent states
  - Precise control of covertness

- One-shot channel reliability and resolvability
  - Existence of codebook that is reliable and does not change covertness much
  - Second-order asymptotic

- Coherent-state QPSK with **decreasing** mean photon number
  - Generated state at Alice's system $\rho_{n,XA} \triangleq \sum_{x\in[\![0,3]\!]} \frac{1}{4} |x\rangle\langle x|_X \otimes \left|u_n e^{j\pi x/2}\right\rangle\left\langle u_n e^{j\pi x/2}\right|_A$
  - Induced state at Willie's system $\rho_{n,XW} \triangleq \sum_{x\in[\![0,3]\!]} \frac{1}{4} |x\rangle\langle x|_X \otimes \rho_{\eta k,W}(\sqrt{1-\eta}u_n e^{j\pi x/2})$
  - **INTUITION:** perturbation in displacement around thermal state
- Perturbation theory for quantum fidelity **[Grace-Guha'21]**

LEMMA: FIDELITY ANALYSIS OF COHERENT-STATE QPSK

By setting $u_n \triangleq \frac{\left(4\eta k(\eta k+1)(\delta-2\kappa)^2\right)^{\frac{1}{4}}}{\sqrt{1-\eta}n^{\frac{1}{4}}}$, we have

$$F\left(\rho_{n,W}^{\otimes n}, \rho_{0,W}^{\otimes n}\right) \geqslant 1-(\delta-2\kappa)^2+\frac{(\delta-2\kappa)^4}{2}.$$

- Make sure existence of a codebook that is both reliable and covert

- **INGREDIENT**
  - Position-based coding and sequential decoding **[Wilde'17]**, **[Oskouei et al.'18]**
  - Convex-split lemma **[Anshu et al.'19]**, **[Khatri et al.'19]**

LEMMA: ONE-SHOT CHANNEL RELIABILITY AND RESOLVABILITY

By choosing fix $\epsilon, \kappa, \gamma_1, \gamma_2, \gamma_3 > 0$, then for a bi-partite state ρ_{XA} and a channel $\mathcal{G} : \rho_{XA} \mapsto \rho_{XBW}$, there exists a coding scheme such that

$$\log M \geqslant \mathbb{D}_{\mathrm{H}}^{\epsilon^2/10-\gamma_1}(\rho_{XB} \,\|\, \rho_X \otimes \rho_B) - \log\left(\frac{4\epsilon^2}{10\gamma_1^2}\right),$$

$$\log MK \leqslant \mathbb{D}_{\max}^{\kappa/2-\gamma_2-\gamma_3}(\rho_{XW} \,\|\, \rho_X \otimes \rho_W) + 2\log\left(\frac{1}{\gamma_2}\right) + \log\left(\frac{8}{\gamma_3^2}\right),$$

$$\mathbb{E}_{\mathcal{C},S}\left\{\mathbb{P}\left(\widehat{W} \neq W|S\right)\right\} \leqslant \frac{\epsilon^2}{10},$$

$$\mathbb{E}_{\mathcal{C}}\left\{\frac{1}{2}\|\hat{\rho}_W - \rho_W\|_1\right\} \leqslant \kappa - \gamma_2,$$

where $\mathcal{C}$ is the codebook.

- Analyze **trace distance** directly with techniques developed in **[Tahmasbi-Bloch'19, Zhang et al.'19, Wang-Bloch'21]**

- Constrained to **coherent-state** codebook
  - A **weaker** converse
  - **No entanglement** within modes
  - Possible to lift this constraint

- Establish lower bound on trace distance related to minimum received **photon number** of codewords
  - Require us to find a simple to analyze yet powerful test for Willie
  - Photon counter

- Show existence of good sub-code resulting in **low covertness metric**

- Obtain upper bound on **covert message size** of good sub-code

- Suboptimal photon counting test POVM $\{T, I - T\}$

$$T \triangleq \sum_{m_1+m_2+\cdots+m_n \geqslant \tau} |m_1\rangle\langle m_2| \otimes |m_2\rangle\langle m_2| \otimes \cdots \otimes |m_n\rangle\langle m_n|$$

  - Since modes are in tensor-product states, it reduces to a classical test $T(m^n) = \mathbb{1}\left\{\sum_{i=1}^{n} m_i \geqslant \tau\right\}$

- Carefully choose threshold and use photon number statistics of displaced thermal states with Berry-Esseen Theorem

LEMMA: LOWER BOUND ON TRACE DISTANCE

By setting $\tau = \dfrac{(1-\eta)N_*}{2} + n\eta k$, we have

$$\begin{aligned}\frac{1}{2}\left\|\hat{\rho}_{W^n} - \rho_{0,W}^{\otimes n}\right\|_1 &\geqslant 1 - \alpha - \beta \\ &\geqslant 1 - 2Q\left(\frac{(1-\eta)N_*}{2\sqrt{n\eta k(\eta k+1)}}\right) - \frac{(1-\eta)^2(1+2\eta k)N_*^2}{4\sqrt{2\pi}n^{3/2}[\eta k(\eta k+1)]^{3/2}} - \frac{B_0+B_1}{\sqrt{n}},\end{aligned}$$

where $N_* \triangleq \min_{m\in\mathcal{M}} \sum_{i=1}^{n} \left|\alpha_i^{(m)}\right|^2$ is minimum photon number of codewords, α is false-alarmed probability, β is missed-detection probability, and B_0, B_1 are some constants.

- **INTUITION:** covertness metric is higher in a set of high-photon-number codewords
- Partition code $\mathcal{C}$ into high-photon-number (bad) sub-code $\mathcal{C}^{(h)}$ and low-photon-number (good) sub-code $\mathcal{C}^{(\ell)}$

> **LEMMA:** EXISTENCE OF GOOD SUB-CODE
>
> For any covert codebook $\mathcal{C}$, let $\lim_{n\to\infty} \gamma_n = 0$. Then there exists a sub-code $\mathcal{C}^{(\ell)}$ such that $\left|\mathcal{C}^{(\ell)}\right| \geqslant \gamma_n |\mathcal{C}|$ and photon-number of codewords $N^{(c)} \leqslant A\sqrt{n}$,
>
> $$A \triangleq \frac{2\sqrt{\eta k(\eta k+1)}}{1-\eta} Q^{-1}\left(\frac{1-\delta}{2} - \frac{\nu^2(1-\eta)^2(1+2\eta k)}{4\sqrt{2\pi n}[\eta k(\eta k+1)]^{3/2}} - \gamma_n\right),$$
>
> where ν depends on channel.

- Characterize upper bound on photon-number of codewords in good sub-code

- Code $\mathcal{C}$ consists of K sub-codes $\mathcal{C}_s$ indexed by key $s \in [\![1, K]\!]$
  - Size of each sub-code is M
  - $\mathcal{C}_s^{(\ell)} \triangleq \mathcal{C}_s \cap \mathcal{C}^{(\ell)}$
  - By pigeonhole principle, $\left|\mathcal{C}_s^{(\ell)}\right| \geqslant \gamma_n M$

- We find an upper bound on $\log \left|\mathcal{C}_s^{(\ell)}\right|$ and therefore apply it to M
  - For bosonic channel, what we need is the bound on photon number we derived earlier
  - By Fano's inequality, Holevo bound, and capacity of bosonic channel

$$\log \left|\mathcal{C}_s^{(\ell)}\right| (1 - \frac{\epsilon_n}{\gamma_n}) - 1 \leqslant nC(\frac{A}{\sqrt{n}}, \eta, k) \leqslant \eta A \sqrt{n} \log \left(1 + \frac{1}{(1-\eta)k}\right)$$

$$\begin{aligned} \liminf_{n\to\infty} \frac{\log M}{\sqrt{n}} &\leqslant \liminf_{n\to\infty} \frac{\log \left|\mathcal{C}_s^{(\ell)}\right| - \log \gamma_n}{\sqrt{n}} \\ &= \frac{2\sqrt{\eta k(\eta k + 1)}}{1-\eta} \eta \log \left(1 + \frac{1}{(1-\eta)k}\right) Q^{-1}\left(\frac{1-\delta}{2}\right) \end{aligned}$$

  - $C(N, \bar{N}, \eta)$ is channel capacity of bosonic channel with photon numbers of two ports $N, \bar{N}$ and transmissivity η

- Gaussian ensemble $\left\{ \frac{1}{\pi s_n} \exp\left(-\frac{|\alpha|^2}{s_n} \right), |\alpha\rangle \langle \alpha| \right\}$ with $s_n = \frac{2\sqrt{\eta k(\eta k+1)}}{1-\eta} Q^{-1}\left(\frac{1-\delta}{2} \right)$ would achieve upper-bound
  - **TECHNICAL DIFFICULTY:** our coding theorem does not support uncountable alphabet
- Direct analysis of QPSK signaling with trace distance is challenging
- To extend converse to general *n*-mode states, consider applying an entanglement-breaking processing channel

$$\rho_W \mapsto \sum_m \mathrm{tr}\left(|m\rangle \langle m| \rho_W \right) \otimes |m\rangle \langle m|_M$$

  - Destroy entanglement between modes
  - Any single-mode Gaussian state is thermal state with displacement and symplectic transform

- **CONCLUSION**
  - Covert capacity of bosonic channel depends on choice of covertness metric
  - Trace distance is ultimate covertness metric carrying operational meaning
  - Characterize secret-key throughput of bosonic covert communications

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